MTS2A3 Komputer & Simulasi

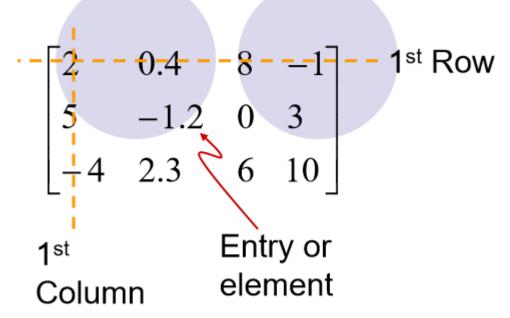
Kuliah 3 – Aljabar Matriks (Review)

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Concepts and Definitions

A matrix is a rectangular array of numbers or functions enclosed in brackets; each such number or function is an entry or element of the matrix.



This matrix has 3 rows and 4 columns, thus is a 3×4 matrix.

Order of matrix: $3 \times 4_{2}$

♦ A vector is a matrix that has only one row or only one column.



Square matrix (number of rows = number of columns)

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad \begin{bmatrix} e^x & -2x & 2e^{-x} \\ x^2 & 3x^3 & x \\ xe^x & -e^{2x} & x+1 \end{bmatrix}$$

General notations $\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$ Capital boldface letter General Array of all entries entry General notations of an $m \times n$ rectangular matrix

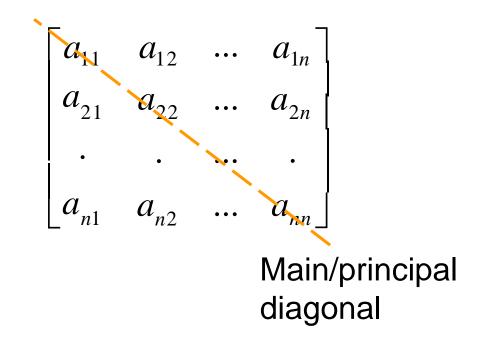
In the double-subscript notation for the entries, the first subscript denotes the row and the second the column in which the given entry stands.

• General notations for vectors $\mathbf{a} = [a_i]$

Lowercase boldface letter

Main/Principal diagonal

Consider an $n \times n$ square matrix



What consists of the main diagonal of a rectangular matrix?

✤ If A = [a_{ij}] is an m × n matrix, the main diagonal of A consists of the entries a_{11} , a_{22} , a_{33} , ...denoted as $diag A = (a_{11}, a_{22}, a_{33}, ..., a_{min(m,n), min(m,n)})$

where min(m,n) stands for the minimum of m and n.

In each of the following matrices, the main diagonal consists of a and b.

$$\begin{bmatrix} a & x \\ y & b \end{bmatrix}; \begin{bmatrix} a & x & p \\ y & b & q \end{bmatrix}; \begin{bmatrix} a & x \\ y & b \\ r & s \end{bmatrix}$$

Special matrices

Triangular matrix - either Lower triangular matrix (entries above main diagonal = 0) or Upper triangular matrix (entries below main diagonal = 0).

$$\mathbf{L} = \begin{bmatrix} 2 & 0 & 0 \\ -4 & 3 & 0 \\ -7 & 4 & 4 \end{bmatrix} \qquad \mathbf{U} = \begin{bmatrix} 2 & 4 & 7 \\ 0 & 3 & -4 \\ 0 & 0 & 4 \end{bmatrix}$$

Diagonal matrix – entries off the main diagonal are all zero.

$$\mathbf{D} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Identity matrix - a square matrix with entries on the main diagonal equal to 1 and all other entries equal to 0.

$$\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad \mathbf{C} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

B and **C** are NOT identity matrices.

$$\mathbf{I} = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 \end{bmatrix}$$

identity matrix of size n x n

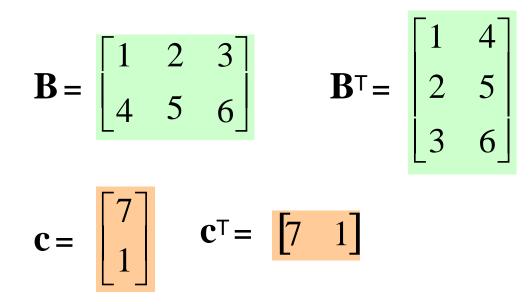
Note that all the above special matrices are square matrices.

Zero/Null matrix - all entries are 0, denoted as $\mathbf{0}$.

Transposition and Symmetry

♦ The transpose A^{T} of an *mxn* matrix $A = [a_{jk}]$ is the *nxm* matrix that has the first row of A as its first column, and second row of A as its second column, and so on.

$$\mathbf{A}^{\mathsf{T}} = \begin{bmatrix} a_{kj} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{21} & \dots & a_{m1} \\ a_{12} & a_{22} & \dots & a_{m2} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ a_{1n} & a_{2n} & \dots & a_{mn} \end{bmatrix}$$
$$\mathbf{A} = \begin{bmatrix} 12 & 15 & 30 \\ 4 & 10 & 12 \\ 6 & 20 & 25 \end{bmatrix} \qquad \mathbf{A}^{\mathsf{T}} = \begin{bmatrix} 12 & 4 & 6 \\ 15 & 10 & 20 \\ 30 & 12 & 25 \end{bmatrix}$$



◆ Equality of matrices: Two matrices $\mathbf{A} = [a_{jk}]$ and $\mathbf{B} = [b_{jk}]$ are equal, written $\mathbf{A} = \mathbf{B}$, if and only if they have the same size and the corresponding entries are equal, that is, $a_{11}=b_{11}$, $a_{12}=b_{12}$, and so on.

Symmetric matrix: A is symmetric if $A^T = A$.

$$\mathbf{A} = \begin{bmatrix} 1 & 4 & 7 \\ 4 & 2 & -4 \\ 7 & -4 & 3 \end{bmatrix} \qquad \mathbf{A}^{\mathsf{T}} = \begin{bmatrix} 1 & 4 & 7 \\ 4 & 2 & -4 \\ 7 & -4 & 3 \end{bmatrix}$$

A must be a square matrix. Why?

Skew-symmetric matrix: B is skew symmetric if $\mathbf{B}^{\mathsf{T}} = -\mathbf{B}$.

$$\mathbf{B} = \begin{bmatrix} 1 & 4 & 7 \\ 4 & 2 & -4 \\ 7 & -4 & 3 \end{bmatrix} \quad \mathbf{B}^{\mathsf{T}} = \begin{bmatrix} 1 & 4 & 7 \\ 4 & 2 & -4 \\ 7 & -4 & 3 \end{bmatrix}$$

B must be a square matrix with main diagonal entries equal to zero. Why?

Matrix Addition, Scalar Multiplication

Addition/subtraction is defined only for matrices of the same size. Their sum/difference is obtained by adding/subtracting the corresponding entries:

 $\mathbf{A} \pm \mathbf{B} = [a_{ij} \pm b_{ij}]$

$$\mathbf{A} = \begin{bmatrix} -4 & 6 & 3 \\ 0 & 1 & 2 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 5 & -1 & 0 \\ 3 & 1 & 0 \end{bmatrix}$$
$$\mathbf{A} + \mathbf{B} = \begin{bmatrix} -4+5 & 6+(-1) & 3+0 \\ 0+3 & 1+1 & 2+0 \end{bmatrix} = \begin{bmatrix} 1 & 5 & 3 \\ 3 & 2 & 2 \end{bmatrix}$$

 $\mathbf{A} - \mathbf{B} = ?$

 \diamond The product of matrix **A** and scalar *c*, written

 $c \mathbf{A}$, is obtained by multiplying each entry in \mathbf{A} by c.

$$\mathbf{A} = \begin{bmatrix} -4 & 6 & 3 \\ 0 & 1 & 2 \end{bmatrix} \quad \text{Then } 3\mathbf{A} = \begin{bmatrix} -12 & 18 & 9 \\ 0 & 3 & 6 \end{bmatrix}$$

The followings are clear:

$$1 A = A$$

(-1) $A = -A$
 $c 0 = 0$
 $0 A = 0$



If
$$\mathbf{A} = \begin{bmatrix} 1 & 0 & -3 \\ 2 & 7 & 3 \end{bmatrix}$$
, and $\mathbf{B} = \begin{bmatrix} 5 & -6 & 1 \\ 8 & 8 & 9 \end{bmatrix}$
then $5\mathbf{A} - 2\mathbf{B} = \begin{bmatrix} -5 & 12 & -17 \\ -6 & 19 & -3 \end{bmatrix}$

3A + 4B = ?

Properties of addition and scalar multiplication

For addition A + B = B + A (commutative law) (U + V) + W = U + (V + W) (associative law) A + 0 = AA + (-A) = 0

For scalar multiplication

$c(\mathbf{A} + \mathbf{B})$	$= c\mathbf{A} + c\mathbf{B}$
$(c+d)\mathbf{A}$	$= c\mathbf{A} + d\mathbf{A}$
$c(d\mathbf{A})$	$= (cd)\mathbf{A}$

For transposition $(\mathbf{A} + \mathbf{B})^{\mathsf{T}}$ $(c\mathbf{A})^{\mathsf{T}}$

$$= \mathbf{A}^{\mathsf{T}} + \mathbf{B}^{\mathsf{T}}$$
$$= \mathscr{C}^{\mathsf{A}} \mathbf{A}^{\mathsf{1}} \mathbf{F}^{\mathsf{Part 1}}$$



If A and B are symmetric, show that A + B is also symmetric.

Proof:

Matrix Multiplication

The product of two matrices B and C (in this order) is defined as

$$\mathbf{A}_{\mathbf{m}\times\mathbf{n}} = \mathbf{B}_{\mathbf{m}\times\mathbf{p}} \mathbf{C}_{\mathbf{p}\times\mathbf{n}}$$

Note the product is only defined (or exist) when the second dimension of ${\bf B}$ (no. of columns) is the same as the first dimension of ${\bf C}$ (no. of rows).

The element of matrix \mathbf{A} is determined as

$$a_{jk} = \sum_{i=1}^{p} b_{ji} c_{ik}, \qquad j = 1 \text{ to m, } k = 1 \text{ to n.}$$

Why is matrix multiplication defined in this way? (Linear Transformation)

$$a_{jk} = \sum_{i=1}^{p} b_{ji} c_{ik} = b_{j1} c_{1k} + b_{j2} c_{2k} + \dots + b_{jp} c_{pk}$$

$$j = 1 \text{ to m, } k = 1 \text{ to n}$$

That is, to get a_{jk} , multiply each entry in the *j*th row of **B** by the corresponding entry in the *k*th column of **C** and then add these *p* products.

$$\begin{bmatrix} b_{11} & b_{12} & \dots & b_{1p} \\ b_{21} & b_{22} & \dots & b_{2p} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ b_{j1} & b_{j2} & \dots & b_{jp} \\ \vdots & \vdots & \dots & \vdots \\ b_{m1} & b_{m2} & \dots & b_{mp} \end{bmatrix} \begin{bmatrix} c_{11} & c_{12} & \dots & c_{1k} & \dots & c_{1n} \\ c_{21} & c_{22} & \dots & c_{2k} & \dots & c_{2n} \\ \vdots & \vdots & \dots & \vdots & \dots & \vdots \\ c_{p1} & c_{p2} & \dots & c_{pk} & \dots & c_{pn} \end{bmatrix} = \begin{bmatrix} \dots & \vdots & \dots \\ \dots & a_{jk} & \dots \\ \dots & \vdots & \dots \end{bmatrix}$$

Multiplication of rows into columns

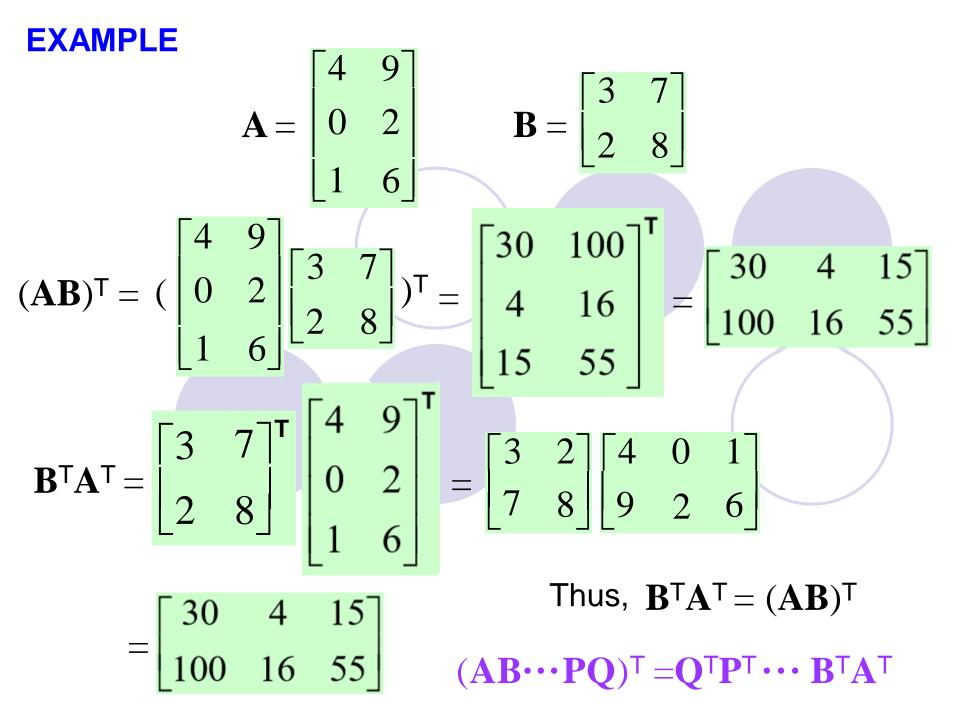
EXAMPLE

$$\mathbf{BC} = \begin{bmatrix} 4 & 3 \\ 7 & 2 \\ 9 & 0 \end{bmatrix} \begin{bmatrix} 2 & 5 \\ 1 & 6 \end{bmatrix} = \begin{bmatrix} 4 \times 2 + 3 \times 1 & 4 \times 5 + 3 \times 6 \\ 7 \times 2 + 2 \times 1 & 7 \times 5 + 2 \times 6 \\ 9 \times 2 + 0 \times 1 & 9 \times 5 + 0 \times 6 \end{bmatrix} = \begin{bmatrix} 11 & 38 \\ 16 & 47 \\ 18 & 45 \end{bmatrix}$$
$$\mathbf{CB} = \begin{bmatrix} 2 & 5 \\ 1 & 6 \end{bmatrix} \begin{bmatrix} 4 & 3 \\ 7 & 2 \\ 9 & 0 \end{bmatrix} = \text{undefined}$$
$$\mathbf{EXAMPLE}$$
$$\mathbf{BC} = \begin{bmatrix} 4 & 3 \\ 7 & 2 \end{bmatrix} \begin{bmatrix} 2 & 5 \\ 1 & 6 \end{bmatrix} = \begin{bmatrix} 11 & 38 \\ 16 & 47 \end{bmatrix}$$
$$\mathbf{CB} = \begin{bmatrix} 2 & 5 \\ 1 & 6 \end{bmatrix} \begin{bmatrix} 4 & 3 \\ 7 & 2 \end{bmatrix} \begin{bmatrix} 2 & 5 \\ 1 & 6 \end{bmatrix} = \begin{bmatrix} 11 & 38 \\ 16 & 47 \end{bmatrix}$$
$$\mathbf{CB} = \begin{bmatrix} 2 & 5 \\ 1 & 6 \end{bmatrix} \begin{bmatrix} 4 & 3 \\ 7 & 2 \end{bmatrix} = \begin{bmatrix} 43 & 16 \\ 46 & 15 \end{bmatrix} \neq \mathbf{BC}$$

Properties of matrix multiplication

- AB is not necessarily equal to BA
- AB = 0 does not necessarily imply
- that $\mathbf{A} = \mathbf{0}, \mathbf{B} = \mathbf{0}, \text{ or } \mathbf{B}\mathbf{A} = \mathbf{0}$
- AC = AD does not necessarily imply that C = D
- AI = A = IA(kA)B = k(AB) = A(kB) = kAB
- A(BC) = (AB)C = ABC
- $\mathbf{A}(\mathbf{B}\mathbf{C}) = (\mathbf{A}\mathbf{B})\mathbf{C} = \mathbf{A}\mathbf{B}\mathbf{C}$
- (A+B)C = AC + BC; C(A+B) = CA + CB
- $(\mathbf{A}\mathbf{B})^{\mathsf{T}} = \mathbf{B}^{\mathsf{T}}\mathbf{A}^{\mathsf{T}} \qquad (\mathbf{A}\mathbf{B}\cdots\mathbf{P}\mathbf{Q})^{\mathsf{T}} = \mathbf{Q}^{\mathsf{T}}\mathbf{P}^{\mathsf{T}}\cdots\mathbf{B}^{\mathsf{T}}\mathbf{A}^{\mathsf{T}}$

$$\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$
$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$$
$$\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 4 & 3 \\ 8 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 1 & 3 \end{bmatrix}$$



Remarks: If the order of the factors in a product of matrices is changed, the product matrix may change (or may not exist at all!). Ignoring this is the source of many errors.

For example when multiplying a matrix equation $\mathbf{B} = \mathbf{C}$ by another matrix \mathbf{A} , care must be taken to multiply \mathbf{B} and \mathbf{C} on the same side by \mathbf{A} .

> **Pre-multiplication** AB = AC**Post-multiplication** BA = CA

Example

If A, B and C are matrices such that AB = I and CA = I, show that B = C.

Solution: