MTS2A3
Komputer & Simulasi

Kuliah 3 – Aljabar Matriks (Review)

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Concepts and Definitions

❖A **matrix** is a rectangular array of numbers or functions enclosed in brackets; each such number or function is an entry or element of the matrix.

$$\begin{bmatrix} 2 & 0.4 & 8 & -1 \\ 5 & -1.2 & 0 & 3 \\ -4 & 2.3 & 6 & 10 \end{bmatrix}$$

1^{st} Row

1^{st} Column

Entry or element

This matrix has 3 rows and 4 columns, thus is a 3×4 matrix.

Order of matrix: 3×4

- A **vector** is a matrix that has only one row or only one column.

$$\begin{bmatrix} x & y & z \end{bmatrix}$$ Row vector

Component

$$\begin{bmatrix} 2 \\ 3 \\ 7 \end{bmatrix}$$ Column vector $$\begin{bmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{bmatrix}$$

- Square matrix (number of rows = number of columns)

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad \begin{bmatrix} e^x & -2x & 2e^{-x} \\ x^2 & 3x^3 & x \\ xe^x & -e^{2x} & x+1 \end{bmatrix}$$

- General notations

$$\mathbf{A} = [a_{jk}] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

Capital boldface letter

General entry

Array of all entries

General notations of an $m \times n$ rectangular matrix

In the double-subscript notation for the entries, the first subscript denotes the row and the second the column in which the given entry stands.

- General notations for vectors $\mathbf{a} = [a_j]$

Lowercase boldface letter

Main/Principal diagonal

Consider an $n \times n$ square matrix

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ . & . & \cdots & . \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

Main/principal diagonal

What consists of the main diagonal of a rectangular matrix?

❖ If $\mathbf{A} = [a_{ij}]$ is an $m \times n$ matrix, the **main diagonal** of $\mathbf{A}$ consists of the entries a_{11}, a_{22}, a_{33}, …denoted as

$$diag\ \mathbf{A} = (a_{11},\ a_{22},\ a_{33},\ \ldots,\ a_{min(m,n),\ min(m,n)})$$

where $min(m,n)$ stands for the minimum of m and n.

In each of the following matrices, the main diagonal consists of a and b.

$$\begin{bmatrix} a & x \\ y & b \end{bmatrix};\quad \begin{bmatrix} a & x & p \\ y & b & q \end{bmatrix};\quad \begin{bmatrix} a & x \\ y & b \\ r & s \end{bmatrix}$$

Special matrices

Triangular matrix - either Lower triangular matrix (entries above main diagonal = 0) or Upper triangular matrix (entries below main diagonal = 0).

$$\mathbf{L} = \begin{bmatrix} 2 & 0 & 0 \\ -4 & 3 & 0 \\ -7 & 4 & 4 \end{bmatrix} \qquad \mathbf{U} = \begin{bmatrix} 2 & 4 & 7 \\ 0 & 3 & -4 \\ 0 & 0 & 4 \end{bmatrix}$$

Diagonal matrix – entries off the main diagonal are all zero.

$$\mathbf{D} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Identity matrix - a square matrix with entries on the main diagonal equal to 1 and all other entries equal to 0.

$$\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad \mathbf{C} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

$\mathbf{B}$ and $\mathbf{C}$ are NOT identity matrices.

$$\mathbf{I} = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 \end{bmatrix}$$

identity matrix of size n x n

Note that all the above special matrices are square matrices.

Zero/Null matrix - all entries are 0, denoted as $\mathbf{0}$.

Transposition and Symmetry

❖ The **transpose** $\mathbf{A}^T$ of an mxn matrix $\mathbf{A} = [a_{jk}]$ is the nxm matrix that has the first row of $\mathbf{A}$ as its first column, and second row of $\mathbf{A}$ as its second column, and so on.

$$\mathbf{A}^T = [a_{kj}] = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ . & . & \cdots & . \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} 12 & 15 & 30 \\ 4 & 10 & 12 \\ 6 & 20 & 25 \end{bmatrix} \qquad \mathbf{A}^T = \begin{bmatrix} 12 & 4 & 6 \\ 15 & 10 & 20 \\ 30 & 12 & 25 \end{bmatrix}$$

$$\mathbf{B} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \qquad \mathbf{B}^{\mathsf{T}} = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

$$\mathbf{c} = \begin{bmatrix} 7 \\ 1 \end{bmatrix} \qquad \mathbf{c}^{\mathsf{T}} = \begin{bmatrix} 7 & 1 \end{bmatrix}$$

❖ **Equality of matrices:** Two matrices $\mathbf{A} = [a_{jk}]$ and $\mathbf{B} = [b_{jk}]$ are equal, written $\mathbf{A} = \mathbf{B}$, if and only if they have the same size and the corresponding entries are equal, that is, $a_{11}=b_{11}$, $a_{12}=b_{12}$, and so on.

❖ **Symmetric matrix:** $\mathbf{A}$ is symmetric if $\mathbf{A}^T = \mathbf{A}$.

$$\mathbf{A} = \begin{bmatrix} 1 & 4 & 7 \\ 4 & 2 & -4 \\ 7 & -4 & 3 \end{bmatrix} \qquad \mathbf{A}^T = \begin{bmatrix} 1 & 4 & 7 \\ 4 & 2 & -4 \\ 7 & -4 & 3 \end{bmatrix}$$

$\mathbf{A}$ *must be a square matrix. Why?*

❖ **Skew-symmetric matrix:** $\mathbf{B}$ is skew symmetric if $\mathbf{B}^T = -\mathbf{B}$.

$$\mathbf{B} = \begin{bmatrix} 1 & 4 & 7 \\ 4 & 2 & -4 \\ 7 & -4 & 3 \end{bmatrix} \qquad \mathbf{B}^T = \begin{bmatrix} 1 & 4 & 7 \\ 4 & 2 & -4 \\ 7 & -4 & 3 \end{bmatrix}$$

$\mathbf{B}$ *must be a square matrix with main diagonal entries equal to zero. Why?*

Matrix Addition, Scalar Multiplication

❖Addition/subtraction is defined only for matrices of the same size. Their sum/difference is obtained by adding/subtracting the corresponding entries:

$$\mathbf{A} \pm \mathbf{B} = [a_{ij} \pm b_{ij}]$$

$$\mathbf{A} = \begin{bmatrix} -4 & 6 & 3 \\ 0 & 1 & 2 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 5 & -1 & 0 \\ 3 & 1 & 0 \end{bmatrix}$$

$$\mathbf{A} + \mathbf{B} = \begin{bmatrix} -4+5 & 6+(-1) & 3+0 \\ 0+3 & 1+1 & 2+0 \end{bmatrix} = \begin{bmatrix} 1 & 5 & 3 \\ 3 & 2 & 2 \end{bmatrix}$$

$$\mathbf{A} - \mathbf{B} = \ ?$$

❖ The product of matrix $\mathbf{A}$ and scalar c, written $c\,\mathbf{A}$, is obtained by multiplying each entry in $\mathbf{A}$ by c.

$$\mathbf{A} = \begin{bmatrix} -4 & 6 & 3 \\ 0 & 1 & 2 \end{bmatrix} \qquad \text{Then } 3\mathbf{A} = \begin{bmatrix} -12 & 18 & 9 \\ 0 & 3 & 6 \end{bmatrix}$$

❖ The followings are clear:

$$1\,\mathbf{A} = \mathbf{A}$$

$$(-1)\,\mathbf{A} = -\mathbf{A}$$

$$c\,\mathbf{0} = \mathbf{0}$$

$$0\,\mathbf{A} = \mathbf{0}$$

Example

If $\mathbf{A} = \begin{bmatrix} 1 & 0 & -3 \\ 2 & 7 & 3 \end{bmatrix}$, and $\mathbf{B} = \begin{bmatrix} 5 & -6 & 1 \\ 8 & 8 & 9 \end{bmatrix}$

then $5\mathbf{A} - 2\mathbf{B} = \begin{bmatrix} -5 & 12 & -17 \\ -6 & 19 & -3 \end{bmatrix}$

$$3\mathbf{A} + 4\mathbf{B} = ?$$

Properties of addition and scalar multiplication

For addition

$$\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$$ (commutative law)

$$(\mathbf{U} + \mathbf{V}) + \mathbf{W} = \mathbf{U} + (\mathbf{V} + \mathbf{W})$$ (associative law)

$$\mathbf{A} + \mathbf{0} = \mathbf{A}$$

$$\mathbf{A} + (-\mathbf{A}) = \mathbf{0}$$

For scalar multiplication

$$c(\mathbf{A} + \mathbf{B}) = c\mathbf{A} + c\mathbf{B}$$

$$(c+d)\,\mathbf{A} = c\mathbf{A} + d\mathbf{A}$$

$$c(d\mathbf{A}) = (cd)\mathbf{A}$$

For transposition

$$(\mathbf{A} + \mathbf{B})^{\mathsf{T}} = \mathbf{A}^{\mathsf{T}} + \mathbf{B}^{\mathsf{T}}$$

$$(c\mathbf{A})^{\mathsf{T}} = c\mathbf{A}^{\mathsf{T}}$$

Example

If $\mathbf{A}$ and $\mathbf{B}$ are symmetric, show that $\mathbf{A}+\mathbf{B}$ is also symmetric.

Proof:

Matrix Multiplication

- The product of two matrices $\mathbf{B}$ and $\mathbf{C}$ (in this order) is defined as

$$\mathbf{A}_{\mathrm{m}\times\mathrm{n}} = \mathbf{B}_{\mathrm{m}\times\mathrm{p}}\,\mathbf{C}_{\mathrm{p}\times\mathrm{n}}$$

Note the product is only defined (or exist) when the second dimension of $\mathbf{B}$ (no. of columns) is the same as the first dimension of $\mathbf{C}$ (no. of rows).

The element of matrix $\mathbf{A}$ is determined as

$$a_{jk} = \sum_{i=1}^{p} b_{ji} c_{ik}\,, \qquad j = 1 \text{ to m},\ k = 1 \text{ to n}.$$

Why is matrix multiplication defined in this way?
(Linear Transformation)

$$a_{jk} = \sum_{i=1}^{p} b_{ji} c_{ik} = b_{j1} c_{1k} + b_{j2} c_{2k} + \cdots + b_{jp} c_{pk}$$

j = 1 to m, k = 1 to n

That is, to get a_{jk}, multiply each entry in the jth row of $\mathbf{B}$ by the corresponding entry in the kth column of $\mathbf{C}$ and then add these p products.

$$\begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1p} \\ b_{21} & b_{22} & \cdots & b_{2p} \\ \cdot & \cdot & \cdots & \cdot \\ b_{j1} & b_{j2} & \cdots & b_{jp} \\ \cdot & \cdot & \cdots & \cdot \\ b_{m1} & b_{m2} & \cdots & b_{mp} \end{bmatrix} \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1k} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2k} & \cdots & c_{2n} \\ \cdot & \cdot & \cdots & \cdot & \cdots & \cdot \\ c_{p1} & c_{p2} & \cdots & c_{pk} & \cdots & c_{pn} \end{bmatrix} = \begin{bmatrix} \cdots & \cdot & \cdots \\ \cdots & a_{jk} & \cdots \\ \cdots & \cdot & \cdots \end{bmatrix}$$

Multiplication of rows into columns

EXAMPLE

$$\mathbf{BC} = \begin{bmatrix} 4 & 3 \\ 7 & 2 \\ 9 & 0 \end{bmatrix} \begin{bmatrix} 2 & 5 \\ 1 & 6 \end{bmatrix} = \begin{bmatrix} 4\times 2+3\times 1 & 4\times 5+3\times 6 \\ 7\times 2+2\times 1 & 7\times 5+2\times 6 \\ 9\times 2+0\times 1 & 9\times 5+0\times 6 \end{bmatrix} = \begin{bmatrix} 11 & 38 \\ 16 & 47 \\ 18 & 45 \end{bmatrix}$$

$$\mathbf{CB} = \begin{bmatrix} 2 & 5 \\ 1 & 6 \end{bmatrix} \begin{bmatrix} 4 & 3 \\ 7 & 2 \\ 9 & 0 \end{bmatrix} = \text{undefined}$$

EXAMPLE

$$\mathbf{BC} = \begin{bmatrix} 4 & 3 \\ 7 & 2 \end{bmatrix} \begin{bmatrix} 2 & 5 \\ 1 & 6 \end{bmatrix} = \begin{bmatrix} 11 & 38 \\ 16 & 47 \end{bmatrix}$$

$$\mathbf{CB} = \begin{bmatrix} 2 & 5 \\ 1 & 6 \end{bmatrix} \begin{bmatrix} 4 & 3 \\ 7 & 2 \end{bmatrix} = \begin{bmatrix} 43 & 16 \\ 46 & 15 \end{bmatrix} \neq \mathbf{BC}$$

Properties of matrix multiplication

$\mathbf{AB}$ is not necessarily equal to $\mathbf{BA}$

$\mathbf{AB} = \mathbf{0}$ does not necessarily imply that $\mathbf{A} = \mathbf{0}$, $\mathbf{B} = \mathbf{0}$, or $\mathbf{BA} = \mathbf{0}$

$\mathbf{AC} = \mathbf{AD}$ does not necessarily imply that $\mathbf{C} = \mathbf{D}$

$\mathbf{AI} = \mathbf{A} = \mathbf{IA}$

$(k\mathbf{A})\mathbf{B} = k(\mathbf{AB}) = \mathbf{A}(k\mathbf{B}) = k\mathbf{AB}$

$\mathbf{A}(\mathbf{BC}) = (\mathbf{AB})\mathbf{C} = \mathbf{ABC}$

$(\mathbf{A}+\mathbf{B})\mathbf{C} = \mathbf{AC} + \mathbf{BC}$; $\mathbf{C}(\mathbf{A}+\mathbf{B}) = \mathbf{CA} + \mathbf{CB}$

$(\mathbf{AB})^{\mathsf{T}} = \mathbf{B}^{\mathsf{T}}\mathbf{A}^{\mathsf{T}}$ $\quad (\mathbf{AB}\cdots\mathbf{PQ})^{\mathsf{T}} = \mathbf{Q}^{\mathsf{T}}\mathbf{P}^{\mathsf{T}}\cdots\mathbf{B}^{\mathsf{T}}\mathbf{A}^{\mathsf{T}}$

$$\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 4 & 3 \\ 8 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 1 & 3 \end{bmatrix}$$

EXAMPLE

$$\mathbf{A} = \begin{bmatrix} 4 & 9 \\ 0 & 2 \\ 1 & 6 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 3 & 7 \\ 2 & 8 \end{bmatrix}$$

$$(\mathbf{AB})^T = \left(\begin{bmatrix} 4 & 9 \\ 0 & 2 \\ 1 & 6 \end{bmatrix} \begin{bmatrix} 3 & 7 \\ 2 & 8 \end{bmatrix} \right)^T = \begin{bmatrix} 30 & 100 \\ 4 & 16 \\ 15 & 55 \end{bmatrix}^T = \begin{bmatrix} 30 & 4 & 15 \\ 100 & 16 & 55 \end{bmatrix}$$

$$\mathbf{B}^T\mathbf{A}^T = \begin{bmatrix} 3 & 7 \\ 2 & 8 \end{bmatrix}^T \begin{bmatrix} 4 & 9 \\ 0 & 2 \\ 1 & 6 \end{bmatrix}^T = \begin{bmatrix} 3 & 2 \\ 7 & 8 \end{bmatrix} \begin{bmatrix} 4 & 0 & 1 \\ 9 & 2 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 30 & 4 & 15 \\ 100 & 16 & 55 \end{bmatrix}$$

Thus, $\mathbf{B}^T\mathbf{A}^T = (\mathbf{AB})^T$

$$(\mathbf{AB}\cdots\mathbf{PQ})^T = \mathbf{Q}^T\mathbf{P}^T \cdots \mathbf{B}^T\mathbf{A}^T$$

❖**Remarks:** If the order of the factors in a product of matrices is changed, the product matrix may change (or may not exist at all!). Ignoring this is the **source of many errors**.

For example when multiplying a matrix equation $\mathbf{B} = \mathbf{C}$ by another matrix $\mathbf{A}$, care must be taken to multiply $\mathbf{B}$ and $\mathbf{C}$ on the same side by $\mathbf{A}$.

Pre-multiplication $\mathbf{AB} = \mathbf{AC}$

Post-multiplication $\mathbf{BA} = \mathbf{CA}$

Example

If $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ are matrices such that $\mathbf{AB} = \mathbf{I}$ and $\mathbf{CA} = \mathbf{I}$, show that $\mathbf{B} = \mathbf{C}$.

Solution: